

FOCK SPACES AND REFINED SEVERI DEGREES

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ABSTRACT. A convex lattice polygon Δ determines a pair (S, L) of a toric surface together with an ample toric line bundle on S . The Severi degree $N^{\Delta, \delta}$ is the number of δ -nodal curves in the complete linear system $|L|$ passing through $\dim |L| - \delta$ general points. Cooper and Pandharipande showed that in the case of $\mathbb{P}^1 \times \mathbb{P}^1$ the Severi degrees can be computed as the matrix elements of an operator on a Fock space. In this note we want to generalize and extend this result in two ways. First we show that it holds more generally for Δ a so called h -transverse lattice polygon. This includes the case of \mathbb{P}^2 and rational ruled surfaces, but also many other, also singular, surfaces. Using a deformed version of the Heisenberg algebra, we extend the result to the refined Severi degrees defined and studied by Göttsche and Shende and by Block and Göttsche. For Δ an h -transverse lattice polygon, one can, following Brugallé and Mikhalkin, replace the count of tropical curves by a count of marked floor diagrams, which are slightly simpler combinatorial objects. We show that these floor diagrams are the Feynman diagrams of certain operators on a Fock space, proving the result.

1. INTRODUCTION

A δ -nodal curve is a reduced (not necessarily irreducible) curve with δ simple nodes as only singularities. The *Severi degrees* $N^{d, \delta}$ are the degrees of the Severi varieties parametrizing δ -nodal plane curves of degree d . Equivalently, $N^{d, \delta}$ is the number of δ -nodal plane curves of degree d through $\frac{(d+3)d}{2} - \delta$ generic points in the complex projective plane \mathbb{P}^2 . More generally, for (S, L) a pair of a surface and line bundle on S , the Severi degree $N^{(S, L), \delta}$ is the number of δ -nodal curves in the complete linear system $|L|$ through $\dim |L| - \delta$ general points in S . In this paper we will deal with the case that S is a toric surface and L a toric line bundle. And slightly contrary to the above we denote $N^{(S, L), \delta}(y)$, the number of of cogenus δ curves in $|L|$ passing through $\dim |L| - \delta$ general points in S , which do not contain a toric boundary divisor as a component, and analogously for the Welschinger invariants below.

The Severi degrees $N^{d, \delta}$ can be computed by the well-known Caporaso-Harris recursion formula [4]. Similar recursion formulas exist for other rational surfaces, in particular for rational ruled surfaces [10]. The Welschinger invariants $W^{(S, L), \delta}(P)$ are the analogues of the Severi degrees and the Gromov-Witten invariants in real algebraic geometry. Under suitable assumptions they are a count of real algebraic curves on a real algebraic surface S , through a real configuration P of $\dim |L| - \delta$ general points. Differently from the Severi degrees, the $W^{(S, L), \delta}(P)$ in general depend

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on the point configuration P . The Severi degrees and the Welschinger invariants of toric surfaces can be computed via tropical geometry.

In [7] and [2] *refined Severi degrees* $N^{d,\delta}(y)$ and $N^{(S,L),\delta}(y)$ are defined for toric surfaces, first for \mathbb{P}^2 on rational ruled surfaces via a Caporaso-Harris type recursion and then in general via tropical geometry. By definition $N^{(S,L),\delta}(1) = N^{(S,L),\delta}$ and $N^{(S,L),\delta}(-1) = W^{(S,L),\delta}(P)$, for some P .

In the recent paper [5] the Severi degrees of $\mathbb{P}^1 \times \mathbb{P}^1$ are expressed as matrix elements of operators on a Fock space, and applications to other rational surfaces are given. This note grew out of an attempt to reprove their results in terms of tropical geometry, and use this approach to extend them to the refined Severi degrees defined in [7], [2]. This is done by using combinatorial gadgets called marked floor diagrams, which have been used in tropical enumerative geometry for some time. It turns out that this leads to a generalization of the results of [5]. There is a dictionary relating expressions in the Heisenberg algebra operators to marked floor diagrams, and this can be used to give formulas for the refined Severi degrees as matrix elements of operators on a Fock space, whenever the refined Severi degrees can be expressed in terms of marked floor diagrams. This includes the refined Severi degrees of \mathbb{P}^2 , rational ruled surfaces, weighted projective spaces, and more generally h -transverse lattice polygons as studied in [1], [3]. Here in the introduction we state our result for \mathbb{P}^2 and rational ruled surfaces Σ_m . We denote by H the hyperplane bundle on \mathbb{P}^2 . On a Hirzebruch surface Σ_m we denote by F the class of fibre of the ruling, E the section with $E^2 = -m$ and $H := E + mF$.

We use the notations about the Heisenberg algebra introduced in §2.1 below: the elements a_k, b_k with $k \in \mathbb{Z}$ are the generators of a Heisenberg algebra \mathcal{H} . The Fock space F is an irreducible representation of \mathcal{H} a vector space with basis consisting of vectors $v_{\alpha,\beta}$, with α, β running through all partitions.

Theorem 1.1. *For $m \in \mathbb{Z}$ let*

$$H_m(t) := \sum_{k>0} b_{-k} b_k + t \sum_{\|\mu\|=\|\nu\|-m} a_{-\mu} a_{\nu}.$$

Then

(1)

$$N^{d,\delta}(y) = \left\langle v_{\emptyset} \left| \text{Coeff}_{t^d} [H_1(t)^{d(d+3)/2-\delta}] \right| v_{(1^d),\emptyset} \right\rangle,$$

(2)

$$N^{(\Sigma_m, cF+dH),\delta}(y) = \left\langle v_{(1^c),\emptyset} \left| \text{Coeff}_{t^d} [H_m(t)^{\binom{d+1}{2}m+dc+d+c-\delta}] \right| v_{(1^{dm+c}),\emptyset} \right\rangle.$$

Roughly what happens is as follows: There is a one to one correspondence between certain monomials in the Heisenberg algebra operators and corresponding vertices of a floor diagram including numbers and weights of the ingoing and outgoing edges. The commutation relations in the Heisenberg algebra then correspond to the different ways how to connect the vertices to form a marked floor diagram, respecting the edges and their multiplicities. Thus the marked floor diagrams become the Feynman

diagrams associated to these monomials in the Heisenberg algebra. The result then follows from a version of Wick's theorem which says that vacuum expectation values of Heisenberg operators on the Fock space can be computed in terms of Feynman diagrams.

This paper is organized as follow: in Section 2 we review the Heisenberg algebra, Fock space, and refined Severi degrees. In Section 3, we state our main result (Theorem 3.5) relating refined Severi degrees and the Fock space and various corollaries. We introduce marked floor diagrams in Section 4. In Section 5, we show that marked floor diagrams are Feynman diagrams and prove a version of Wick's theorem implying our results of Section 3.

2. BACKGROUND

2.1. The Heisenberg algebra and the Fock space. We review the Fock space introduced in [5], changing some of the notations and conventions from there and introducing a y -deformation. For $n \in \mathbb{Z}$ we define the *quantum number*

$$[n]_y := \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}} = y^{(n-1)/2} + \dots + y^{-(n-1)/2}.$$

Note that $[n]_1 = n$. We consider the following y -deformed version \mathcal{H} of the Heisenberg algebra modeled on the hyperbolic lattice. \mathcal{H} is the Lie algebra over $\mathbb{Q}[y^{\pm 1/2}]$ generated by operators a_n, b_n , for all $n \in \mathbb{Z}$, with the commutation relations

$$(2.1) \quad [a_n, a_m] = [b_n, b_m] = 0, \quad [a_n, b_m] = [n]_y \delta_{n,-m}.$$

The a_{-n}, b_{-n} with $n > 0$ are called *creation operators*, the a_n, b_n with $n > 0$ are called *annihilation operators*. We put $a_0 = b_0 := 0$.

The *Fock space* F is the free $\mathbb{Q}[y^{\pm 1/2}]$ -module generated by the creation operators a_{-n}, b_{-n} , ($n > 0$) acting on the so-called *vacuum vector* $v_\emptyset \in F$. F is an \mathcal{H} -module by requiring that $a_n v_\emptyset = b_n v_\emptyset = 0$ for all $n > 0$. F has therefore a $\mathbb{Q}[y^{\pm 1/2}]$ -basis parametrized by pairs of partitions.

We write partitions as $\mu = (1^{\mu_1}, 2^{\mu_2}, \dots)$, where μ_i is the number of times i occurs in μ . We denote by $\|\mu\| := \sum_i i\mu_i$ the number partitioned by μ and $|\mu| = \sum_i \mu_i$ the length of the partition. We denote \emptyset the empty partition of 0.

For a partition μ , we define operators

$$(2.2) \quad a_\mu := \prod_i \frac{(a_i)^{\mu_i}}{\mu_i!}, \quad a_{-\mu} := \prod_i \frac{(a_{-i})^{\mu_i}}{\mu_i!}, \quad b_\mu := \prod_i \frac{(b_i)^{\mu_i}}{\mu_i!}, \quad b_{-\mu} := \prod_i \frac{(b_{-i})^{\mu_i}}{\mu_i!} \in \mathcal{H}.$$

In particular $a_\emptyset = a_{-\emptyset} = 1$, and similar for the b_μ . For partitions μ, ν we define $v_{\mu,\nu} := a_{-\mu} b_{-\nu} v_\emptyset$. Then the $v_{\mu,\nu}$ form a $\mathbb{Q}[y^{\pm 1/2}]$ -basis of F . A $\mathbb{Q}[y^{\pm 1/2}]$ -bilinear inner product $\langle | \rangle$ is defined by $\langle v_\emptyset | v_\emptyset \rangle := 1$ and the condition that a_n is adjoint to a_{-n} and b_n to b_{-n} . Explicitely this gives

$$(2.3) \quad \langle v_{\mu,\nu} | v_{\mu',\nu'} \rangle = \left(\prod_i \frac{([i]_y)^{\mu_i}}{\mu_i!} \right) \left(\prod_j \frac{([j]_y)^{\nu_j}}{\nu_j!} \right) \delta_{\mu,\nu'} \delta_{\nu,\mu'}.$$

We also write $\langle \alpha | A | \beta \rangle := \langle \alpha | A \beta \rangle$ for $\alpha, \beta \in F$, $A \in \mathcal{H}$. If $A = \sum_n A_n t^n$ is an operator in $\mathcal{H}[[t]]$, we write $\langle v_{\mu, \nu} | A | v_{\mu', \nu'} \rangle$ for the matrix element $\sum_n \langle v_{\mu, \nu} | A_n | v_{\mu', \nu'} \rangle t^n$. We write $\langle A \rangle$ for the *vacuum expectation value* $\langle v_\emptyset | A | v_\emptyset \rangle$.

2.2. Refined Severi degrees. We briefly review the definition of the refined Severi degrees from [2], more details can be found there.

A *lattice polygon* $\Delta \subset \mathbb{R}^2$ is a polygon with vertices of integer coordinates. The *lattice length* of an edge e of Δ is $\#e \cap \mathbb{Z}^2 - 1$. Δ is a closed subset of \mathbb{R}^2 . We denote by $\text{int}(\Delta)$, $\partial(\Delta)$ its interior and its boundary. To a convex lattice polygon Δ one can associate a pair $S(\Delta)$, $L(\Delta)$ of a toric surface and a toric line bundle on $S(\Delta)$. The toric surface is defined by the fan given by the outer normal vectors of Δ . We have $\dim H^0(S(\Delta), L(\Delta)) = \#(\Delta \cap \mathbb{Z}^2)$. The arithmetic genus of a curve in $|L(\Delta)|$ is $g(\Delta) = \#(\text{int}(\Delta) \cap \mathbb{Z}^2)$.

Let Δ be a lattice polygon in \mathbb{R}^2 . A non-zero vector $u \in \mathbb{Z}^2$ is *primitive* if its entries are coprime.

Definition 2.1. A *tropical curve of degree Δ* is a continuous map $h : C \rightarrow \mathbb{R}^2$ satisfying:

- (1) C is a abstract tropical curve (essentially a metric graph), possibly with multiple components.
- (2) $h(C)$ is a one-dimensional polyhedral complex with edges of rational slope, and non-negative integer weights $w(e)$ on all edges e , such that each vertex V of C is balanced, that is

$$\sum_{e: V \in \partial e} w(e) \cdot v(V, e) = 0,$$

where $v(V, e) \in \mathbb{Z}^2$ is the primitive vector starting at V in direction e .

- (3) For each primitive vector $u \in \mathbb{Z}^2$, the total weight of the unbounded edges in direction u equals the lattice length of an edge of $\partial\Delta$ with outer normal vector u (if there is no such edge, we require the total weight to be zero).

A tropical curve (C, h) defines a *dual subdivision* of Δ . (The edges of this subdivision are orthogonal to the edges e of $h(C)$ and have lattice length $w(e)$.) Each 3-valent vertex v of $h(C)$ corresponds to a triangle Δ_v of the dual subdivision. We say that (C, h) is *simple* if all vertices of C are 3-valent, the self-intersections of h are disjoint from vertices, and the inverse image under h of self-intersection points consists of exactly two points of C . The *number of nodes* of (C, h) is the number of parallelograms of the dual subdivision if (C, h) is simple. The lattice area $\text{Area}(-)$ of a lattice polygon is twice its Euclidian area.

The *refined multiplicity* of a simple tropical curve (C, h) is

$$(2.4) \quad \text{mult}(C, h; y) = \prod_v [\text{Area}(\Delta_v)]_y,$$

the product running over the 3-valent vertices of (C, h) .

We now define the tropical refinement of Severi degrees. We require the configuration of tropical points to be in *tropically generic position*; the precise definition is

given in [9, Definition 4.7]. Roughly, tropically generic means there are no tropical curves of unexpectedly small degree passing through the points.

Definition 2.2. The *refined Severi degree* $N^{\Delta, \delta}(y)$ is

$$(2.5) \quad N^{\Delta, \delta}(y) := \sum_{(C, h)} \text{mult}(C, h; y),$$

where the sum is over all δ -nodal tropical curves (C, h) of degree Δ passing through $|\Delta \cap \mathbb{Z}^2| - 1 - \delta$ tropically generic points.

By [8, Theorem 1] and [2, Theorem 7.3], $N^{\Delta, \delta}(y)$ is independent of the generic point configuration.

There is a relative notion of refined Severi degrees if Δ has at least one horizontal edge as is the case for \mathbb{P}^2 , Σ_m , and $\mathbb{P}(1, 1, m)$. Again, we are brief, see Definitions 7.1 and 7.2 of [2] for details. We now assume that Δ has a horizontal edge at the bottom (i.e., with outer normal $(0, -1)$) of lattice length d^b . By a *sequence* we mean a collection $\alpha = (\alpha_1, \alpha_2, \dots)$ of nonnegative integers, almost all of which are zero. For two sequences α, β we define $\alpha! := \prod_i \alpha_i!$, $|\alpha| = \sum_i \alpha_i$, $\|\alpha\| = \sum_i i\alpha_i$, and $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots)$. Note that a sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ can be identified with the corresponding partition $(1^{\alpha_1}, 2^{\alpha_2}, \dots)$ and with this identification the definitions of $\|\alpha\|$ and $|\alpha|$ correspond.

Let α and β be two sequences with $\|\alpha\| + \|\beta\| = d^b$, let D be a horizontal line very far below, and let Π be a tropically generic point configuration of $|\Delta \cap \mathbb{Z}^2| - 1 - \delta - \|\alpha\| - \|\beta\| + |\alpha| + |\beta|$ points with precisely $|\alpha|$ points on D . A tropical curve C passing through Π is (α, β) -*tangent to D* if precisely $\alpha_i + \beta_i$ unbounded edges of C are orthogonal to and intersect D and have multiplicity i and, further, α_i of the edges pass through $\Pi \cap D$.

Definition 2.3. The *refined relative Severi degree* $N^{\Delta, \delta}(\alpha, \beta)(y)$ is the number of δ -nodal tropical curves C of degree Δ passing through Π that are (α, β) -tangent to D , counted with multiplicity

$$\text{mult}_{\alpha, \beta}(C; y) = \frac{1}{\prod_{i \geq 1} ([i]_y)^{\alpha_i}} \cdot \text{mult}(C; y).$$

2.3. h -transverse lattice polygons. We recall the definition of h -transverse lattice polygons from [1], with slightly different notations.

Definition 2.4. For us a *multiset* of integers is a tuple $(r_1^{i_1}, \dots, r_s^{i_s})$, where the r_j are an ascending sequence of integers and the i_j are positive integers. Thus they are the same as the maps $\mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ with finite support. For a multiset $r = (r_1^{i_1}, \dots, r_s^{i_s})$, we put $|r| = \sum_{j=1}^s i_j$ and $\|r\| = \sum_{j=1}^s i_j r_j$. An *ordering* of r is a sequence $(a_1, a_2, \dots, a_{|r|})$ of integers, with $i_j = \#\{k \mid a_k = r_j\}$ for all j . For instance the orderings of $(1^2, 2)$ are $(1, 1, 2)$, $(1, 2, 1)$, $(2, 1, 1)$.

Definition 2.5. A convex lattice polygon $\Delta \subset \mathbb{R}^2$ is called h -transverse, if every edge has slope 0, ∞ or $1/k$ for some integer k .

Now let Δ be an h -transverse convex lattice polygon. Let d_Δ^t and d_Δ^b be the lengths of the top and bottom edges of Δ (and 0 if they do not exist). Let the slopes of

outward normal vectors of the edges on the right hand side of Δ be (in ascending order) r_1, \dots, r_s and let their lattice lengths be i_1, \dots, i_s . Similarly let the slopes of the outward normal vectors on the left hand side be (in ascending order) l_1, \dots, l_t and let their lattice lengths be j_1, \dots, j_t . Then

$$r_\Delta := (r_1^{i_1}, \dots, r_s^{i_s}), \quad l_\Delta := (l_1^{j_1}, \dots, l_t^{j_t})$$

are multisets with $d_\Delta^b + \|l_\Delta\| = d_\Delta^t + \|r_\Delta\|$, and $h_\Delta := |l_\Delta| = |r_\Delta|$ is the height of Δ .

Conversely, given $d^b, d^t \in \mathbb{Z}_{\geq 0}$, $r, l : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ with finite support and $|l| = |r|$, such that $d^t + \|r\| = d^b + \|l\|$, there is an h -transverse lattice polygon Δ such that $d^b = d_\Delta^b$, $d^t = d_\Delta^t$, $l = l_\Delta$, $r = r_\Delta$.

Example 2.6. We list some examples of toric surfaces and line bundles corresponding to h -transverse polygons.

- (1) For $S = \mathbb{P}^2$, $L = \mathcal{O}(d)$, we have $d_\Delta^t = 0$, $d_\Delta^b = d$, $l_\Delta = (0^d)$ and $r_\Delta = (1^d)$.
- (2) For $S = \Sigma_m$ a rational ruled surface, let F be the class of a fibre of the ruling; let E be the section with self intersection $-m$; Let $H = E + mF$. For $L = dH + cF$ we have $d_\Delta^t = c$, $d^b = c + dm$, $l_\Delta = (0^d)$, $r_\Delta = (m^d)$.
- (3) For weighted projective space $\mathbb{P}(1, 1, m)$, and $L = dH$ with H the hyperplane bundle with $H^2 = m$, we have $d_\Delta^t = 0$, $d_\Delta^b = dm$, $l_\Delta = (0^d)$ and $r_\Delta = (m^d)$. As dH on Σ_m is the pullback of the line bundle with the same name on $\mathbb{P}(1, 1, m)$, this in fact is the case $c = 0$ of the previous example.
- (4) For a weighted projective space $\mathbb{P}(1, m-1, m)$ and $L = dH$ where H is the hyperplane bundle with $H^2 = m(m-1)$, we have $d_\Delta^t = 0$, $d_\Delta^b = 0$, $l_\Delta = (0^{dm})$, $r_\Delta = ((-1)^{d(m-1)}, (m-1)^d)$.

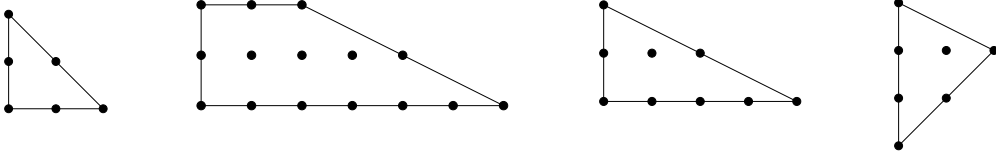


FIGURE 1. The lattice polygons of $(\mathbb{P}^2, 2H)$, $(\Sigma_2, 2F + 2H)$, $(\mathbb{P}(1, 1, 2), 2H)$, and $\mathbb{P}(1, 2, 3), H)$.

3. MAIN THEOREMS

We first state our main result for general h -transverse lattice polygons, relating the refined Severi degrees to matrix elements of operators in the Heisenberg algebra \mathcal{H} . The general formula is a bit complicated, but as corollaries we get somewhat more attractive formulas for the surfaces of Example 2.6.

Remark 3.1. The Fock space F has a grading $F = \bigoplus_{n \geq 0} F_n$ with F_n the span of the $v_{\mu, \nu}$ with $\|\mu\| + \|\nu\| = n$. We denote by \widehat{F} the completion of F with respect to this grading, i.e. elements of \widehat{F} are possibly infinite sums $\sum_{n \in \mathbb{Z}_{\geq 0}} v_n$, with $v_n \in F_n$.

Then \mathcal{H} is a graded algebra: $\mathcal{H} = \sum_{n \in \mathbb{Z}} \mathcal{H}_n$. This grading is defined by giving degree $n \in \mathbb{Z}$ to a_{-n} and b_{-n} . It coincides with the degree as operators on F : elements of \mathcal{H}_n send F_m to F_{m+n} . We denote $\widehat{\mathcal{H}}$ the set of linear maps $f : \widehat{F} \rightarrow \widehat{F}$, which are expressible as possibly infinite sums $f = \sum_{n \geq 0} h_n$ with $h_n \in \mathcal{H}$.

Notation 3.2. Let T_i , $i \in \mathbb{Z}$, be noncommuting variables (with no relations). For a finite sequence $I = (i_1, \dots, i_n)$ of integers, we put $T^I := T_{i_1} T_{i_2} \dots T_{i_n}$. For a ring R , let $R\{T\}$ be the set of finite linear combinations $\sum_I a_I T^I$, with I running through finite sequences of integers with coefficientwise addition, and multiplication defined by concatenation:

$$(T_{i_1} \dots T_{i_n})(T_{j_1} \dots T_{j_m}) = T_{i_1} \dots T_{i_n} T_{j_1} \dots T_{j_m}.$$

For $M = \sum_I a_I T^I \in R\{T\}$, with the I distinct, we put $\text{Coeff}_{T^I}[M] = a_I$.

We define an operator on the Fock space F by

$$H(T) := \sum_{k > 0} b_{-k} b_k + \sum_{\mu, \nu} a_{-\mu} a_{\nu} T_{\|\nu\| - \|\mu\|} \in \widehat{\mathcal{H}}\{T\},$$

where μ, ν run through all partitions. For sequences $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$, as usual $I - J = (i_1 - j_1, \dots, i_n - j_n)$.

Notation 3.3. In the future we will write $\#\Delta$ instead of $\#(\Delta \cap \mathbb{Z}^2)$ for the lattice points in a convex lattice polygon. Below we will usually write d^b and d^t instead of d_{Δ}^b , d_{Δ}^t , when this does not lead to confusion.

Notation 3.4. We write $I_y^{\alpha+\beta} = \prod_i [i]_y^{\alpha_i + \beta_i}$ for finite sequences α and β .

The main theorem of this paper describes the refined Severi degrees of h -transversal lattice polygons in terms of matrix elements of operators on Fock space.

Theorem 3.5. *Let Δ be an h -transverse lattice polygon.*

$$(3.1) \quad N^{\Delta, \delta}(y) = \left\langle v_{(1^{d^t}), \emptyset} \left| \sum_{R, L} \text{Coeff}_{T^{R-L}} [H(T)^{\#\Delta - \delta - 1}] \right| v_{(1^{d^b}), \emptyset} \right\rangle,$$

$$(3.2) \quad N^{\Delta, \delta}(\alpha, \beta)(y) = \frac{\alpha!}{I_y^{\alpha+\beta}} \left\langle v_{(1^{d^t}), \emptyset} \left| \sum_{R, L} \text{Coeff}_{T^{R-L}} [H(T)^{\#\Delta - \delta - 1 - d^b + |\beta|}] \right| v_{\beta, \alpha} \right\rangle.$$

Here R and L run through the orderings of r_{Δ} and l_{Δ} respectively, and in the second formula α, β are partitions with $\|\alpha\| + \|\beta\| = d^b$.

If Δ has only one left direction, i.e. we can write $l_{\Delta} = (l^{h_{\Delta}})$ for some $l \in \mathbb{Z}$, the formula simplifies, and we do not need to use noncommutative variables anymore. Let $(t_i)_{i \in \mathbb{Z}}$ be commuting variables. For a multiset $I = (l_1^{i_1}, \dots, l_s^{i_s})$ of integers, we put $\mathbf{t}^I := t_{l_1}^{i_1} \dots t_{l_s}^{i_s}$. For a ring R we write $R[\mathbf{t}]$ for the ring of polynomials series in the $(t_i)_{i \in \mathbb{Z}}$. We define an operator on the Fock space F by

$$H(\mathbf{t}) := \sum_{k > 0} b_{-k} b_k + \sum_{\mu, \nu} a_{-\mu} a_{\nu} t_{\|\nu\| - \|\mu\|} \in \widehat{\mathcal{H}}[\mathbf{t}],$$

where μ, ν run through all partitions. If $r_{\Delta} = (r_1^{n_1}, \dots, r_s^{n_s})$, we write $(r_{\Delta} - l) := ((r_1 - l)^{n_1}, \dots, (r_s - l)^{n_s})$

Corollary 3.6. *Let Δ be an h -transverse lattice polygon with $l_\Delta = (l^{h_\Delta})$. Then*

$$N^{\Delta, \delta}(y) = \left\langle v_{(1^{d^t}), \emptyset} \left| \text{Coeff}_{\mathbf{t}^{(r_\Delta - l)}} [H(\mathbf{t})^{\# \Delta - \delta - 1}] \right| v_{(1^{d^b}), \emptyset} \right\rangle.$$

$$N^{\Delta, \delta}(\alpha, \beta)(y) = \frac{\alpha!}{I_y^{\alpha + \beta}} \left\langle v_{(1^{d^t}), \emptyset} \left| \text{Coeff}_{\mathbf{t}^{(r_\Delta - l)}} [H(\mathbf{t})^{\# \Delta - \delta - 1 - d^b + |\beta|}] \right| v_{\beta, \alpha} \right\rangle,$$

where α, β are partitions with $\|\alpha\| + \|\beta\| = d^b$.

In Example 2.6 the assumptions of Corollary 3.6 are fulfilled.

For an integer m let

$$H_m(t) := \sum_{k > 0} b_{-k} b_k + t \sum_{\|\mu\| = \|\nu\| - m} a_{-\mu} a_\nu$$

Corollary 3.7. (1) *For \mathbb{P}^2 we have*

$$N^{d, \delta}(y) = \left\langle v_\emptyset \left| \text{Coeff}_{t^d} [H_1(t)^{d(d+3)/2 - \delta}] \right| v_{(1^d), \emptyset} \right\rangle,$$

$$N^{d, \delta}(\alpha, \beta)(y) = \frac{\alpha!}{I_y^{\alpha + \beta}} \left\langle v_\emptyset \left| \text{Coeff}_{t^d} [H_1(t)^{d(d+3)/2 - \delta - d + |\beta|}] \right| v_{\beta, \alpha} \right\rangle,$$

for α, β partitions with $\|\alpha\| + \|\beta\| = d$.

(2) *In the case of rational ruled surfaces Σ_m and the weighted projective space $\mathbb{P}(1, 1, m)$, we get*

$$N^{(\Sigma_m, cF + dH), \delta}(y) = \left\langle v_{(1^c), \emptyset} \left| \text{Coeff}_{t^d} [H_m(t)^{\binom{d+1}{2}m + cd + c + d - \delta}] \right| v_{(1^{dm+c}), \emptyset} \right\rangle,$$

$$N^{(\Sigma_m, cF + dH), \delta}(\alpha, \beta)(y) = \frac{\alpha!}{I_y^{\alpha + \beta}} \left\langle v_{(1^c), \emptyset} \left| \text{Coeff}_{t^d} [H_m(t)^{\binom{d+1}{2}m + cd + c + d - \delta - \|\alpha\| - \|\beta\| + |\beta|}] \right| v_{\beta, \alpha} \right\rangle,$$

$$N^{(\mathbb{P}(1, 1, m), dH), \delta}(y) = \left\langle v_\emptyset \left| \text{Coeff}_{t^d} [H_m(t)^{\frac{m}{2}d^2 + (\frac{m}{2} + 1)d - \delta}] \right| v_{(1^{dm}), \emptyset} \right\rangle,$$

$$N^{(\mathbb{P}(1, 1, m), dH), \delta}(\alpha, \beta)(y) = \frac{\alpha!}{I_y^{\alpha + \beta}} \left\langle v_\emptyset \left| \text{Coeff}_{t^d} [H_m(t)^{\frac{m}{2}d^2 + (\frac{m}{2} + 1)d - \delta - \|\alpha\| - \|\beta\| + |\beta|}] \right| v_{\beta, \alpha} \right\rangle.$$

(3) *Let*

$$G_m(t) := \sum_{k > 0} b_{-k} b_k + \sum_{\|\mu\| = \|\nu\| + 1} a_{-\mu} a_\nu + t \sum_{\|\mu\| = \|\nu\| - m} a_{-\mu} a_\nu.$$

Then for the surface $\mathbb{P}(1, (m-1), m)$ we get

$$N^{(\mathbb{P}(1, (m-1), m), dH), \delta}(y) = \left\langle \text{Coeff}_{t^d} [G_{m-1}(t)^{\binom{m}{2}d^2 + md - \delta}] \right\rangle.$$

Remark 3.8. (1) By definition $\text{Coeff}_{t^d} H_m(t)^N$ has degree $-dm$ with respect to the grading on F . Thus $\langle v_{\alpha', \beta'} | \text{Coeff}_{t^d} H_m(t)^N | v_{\alpha, \beta} \rangle = 0$ unless $\|\alpha\| + \|\beta\| - \|\alpha'\| - \|\beta'\| = dm$. Therefore the formulas in (1) and (2) (but not (3)) of Corollary 3.7 are also true without taking the coefficient of t^d .

- (2) In the completion \widehat{F} of F , we have the easy identity $\exp(a_{-1})v_\emptyset = \sum_{m \geq 0} v_{((1^m), \emptyset)}$. Therefore the argument of (1) shows also that

$$\begin{aligned} \left\langle v_{(1^c), \emptyset} \left| \text{Coeff}_{t^d} H_m(t)^N \right| v_{(1^c + dm), \emptyset} \right\rangle &= \sum_{n \geq 0} \left\langle v_{(1^c), \emptyset} \left| \text{Coeff}_{t^d} H_m(t)^N \right| v_{(1^n), \emptyset} \right\rangle \\ &= \left\langle v_{(1^c), \emptyset} \left| \text{Coeff}_{t^d} H_m(t)^N \right| \exp(a_{-1})v_\emptyset \right\rangle. \end{aligned}$$

Corollary 3.9. *One can organize the above results into generating functions:*

- (1) For \mathbb{P}^2 we have

$$\sum_{d \geq 0} \sum_{\delta \geq 0} \frac{t^d q^{d(d+3)/2 - \delta}}{(d(d+3)/2 - \delta)!} N^{d, \delta}(y) = \langle \exp(qH_1(t)) \exp(a_{-1}) \rangle.$$

- (2) On Σ_m and $\mathbb{P}(1, 1, m)$ we get

$$\begin{aligned} \sum_{c \geq 0} \sum_{d \geq 0} \sum_{\delta \geq 0} \frac{s^c t^d q^{\binom{d+1}{2}m + cd + c + d - \delta}}{(\binom{d+1}{2}m + cd + c + d - \delta)!} N^{(\Sigma_m, cF + dH), \delta}(y) &= \langle \exp(a_1 s) \exp(qH_m(t)) \exp(a_{-1}) \rangle, \\ \sum_{d \geq 0} \sum_{\delta \geq 0} \frac{t^d q^{\frac{m}{2}d^2 + (\frac{m}{2} + 1)d - \delta}}{(\frac{m}{2}d^2 + (\frac{m}{2} + 1)d - \delta)!} N^{(\mathbb{P}(1, 1, m), dH), \delta}(y) &= \langle \exp(qH_m(t)) \exp(a_{-1}) \rangle. \end{aligned}$$

- (3) On $\mathbb{P}(1, m-1, m)$ we get

$$\sum_{d \geq 0} \sum_{\delta \geq 0} \frac{t^d q^{\binom{m}{2}d^2 + md - \delta}}{(\binom{m}{2}d^2 + md - \delta)!} N^{(\mathbb{P}(1, (m-1), m), dH), \delta}(y) = \langle \exp(qG_{m-1}(t)) \rangle.$$

Theorem 3.5 will be proven in §5. In the rest of this section we will deduce Corollary 3.6, Corollary 3.7, Corollary 3.9 from Theorem 3.5.

Proof of Corollary 3.6. The map $T_i \mapsto t_i$ gives a homomorphism $\phi : \widehat{\mathcal{H}}\{T\} \rightarrow \widehat{\mathcal{H}}[\mathbf{t}]$. Clearly $\phi(H(T)^N) = H(\mathbf{t})^N$ for all N . It is also clear by definition that we can write $H(T)^N = \sum_R h_R T^R$, where the coefficient $h_R \in \widehat{\mathcal{H}}$ does not depend on the sequence R , but only on the multiset of which R is an ordering. Thus we can write

$$H(T)^N = \sum_r \sum_R h_r T^R, \quad H(\mathbf{t})^N = \sum_r \left(\sum_R h_r \right) \mathbf{t}^r,$$

where the outer sums are over multisets r of integers and the inner sums are over the orderings R of r , and the h_r are elements of $\widehat{\mathcal{H}}$. Thus

$$(3.3) \quad \sum_R \text{Coeff}_{T^R}[H(T)^N] = \sum_R h_r = \text{Coeff}_{\mathbf{t}^r}[H(\mathbf{t})^N],$$

where again the sums are over the orderings R of r .

We are assuming $l_\Delta = (l^{h_\Delta})$. Thus l_Δ has only the unique ordering $L := (l, \dots, l)$. Thus $R = (r_1, \dots, r_{h_\Delta}) \mapsto R - L := (r_1 - l, \dots, r_{h_\Delta} - l)$ is a bijection of the orderings

of r_Δ to those of $(r_\Delta - l)$. Therefore for any N

$$\sum_{R,L} \text{Coeff}_{T^{R-L}} [H(T)^N] = \sum_{R'} \text{Coeff}_{T^{R'}} [H(T)^N] = \text{Coeff}_{\mathbf{t}^{(r_\Delta - l)}} [H(\mathbf{t})^N],$$

where the first sums are over the reorderings of r_Δ and l_Δ , the second sum is over the reorderings of $(r_\Delta - l)$, and the last identity is by (3.3). Thus Corollary 3.6 follows from Theorem 3.5. \square

Proof of Corollary 3.7. In all the cases we have $l = 0$, thus $(r_\Delta - l) := r_\Delta = (r_1^{n_1}, \dots, r_s^{n_s})$. Then the right hand side of the formulas of Corollary 3.6 does not change if we set $t_i = 0$ whenever $i \notin \{r_1, \dots, r_s\}$. By Example 2.6 we have the following:

(1) For $S = \mathbb{P}^2$, $L = dH$, we have $r_\Delta = (1^d)$, $d_\Delta^t = 0$, $d_\Delta^b = d$ and it is easy to see that $\#\Delta - 1 = d(d+3)/2$. Thus the formulas follow at once by setting $t_i := 0$ for $i \neq 1$, $t_1 := t$ in Corollary 3.6.

(2) For $S = \Sigma_m$, $L = cF + dH$, we have $r_\Delta = (m^d)$, $d_\Delta^t = c$, $d_\Delta^b = c + md$ and it is easy to see that $\#\Delta - 1 = \binom{d+1}{2}m + cd + c + d$. Thus the formula follows by setting $t_i := 0$ for $i \neq m$, $t_m := t$ in Corollary 3.6. The formula for $\mathbb{P}(1, 1, m)$ follows by setting $c = 0$.

(3) For $S = \mathbb{P}(1, m-1, m)$, $L = dH$ we have $r_\Delta = ((-1)^{(m-1)d}, (m-1)^d)$, $d_\Delta^t = d_\Delta^b = 0$. Dividing Δ horizontally into two triangles, congruent to the polygons for $(\mathbb{P}^2, d(m-1)H)$ and for $(\mathbb{P}(1, 1, m-1), dH)$, we compute $\#\Delta - 1 = \binom{m}{2}d^2 + md$. Thus setting $t_i := 0$ for $i \notin \{-1, (m-1)\}$, $t_{-1} := s$, $t_{m-1} := t$ in Corollary 3.6, we get

$$(3.4) \quad N^{(\mathbb{P}(1, (m-1), m), dH), \delta}(y) = \left\langle \text{Coeff}_{s^{(m-1)d}t^d} \left[G_{m-1}(s, t) \binom{m}{2} d^2 + md - \delta \right] \right\rangle,$$

with

$$G_m(s, t) := \sum_{k \geq 0} b_{-k} b_k + s \sum_{\|\mu\| = \|\nu\| + 1} a_{-\mu} a_\nu + t \sum_{\|\mu\| = \|\nu\| - m} a_{-\mu} a_\nu.$$

Note however that the coefficient of s of $G_{m-1}(s, t)$ has degree 1 and the coefficient of t has degree $-(m-1)$. Thus $\langle \text{Coeff}_{s^n t^d} [G_{m-1}(s, t)] \rangle = 0$ unless $n = (m-1)d$, and thus (3.4) remains true if we put $s = 1$. \square

Proof of Corollary 3.9. (1) By Corollary 3.7 and Remark 3.8 we have

$$N^{d, \delta}(y) = \langle v_\emptyset | H_1(t)^{d(d+3)/2 - \delta} | \exp(a_{-1}) v_\emptyset \rangle$$

for all $\delta \in \mathbb{Z}$, and both sides of this equation vanish if $d(d+3)/2 - \delta < 0$. Writing $n := d(d+3)/2 - \delta$, we get therefore

$$\begin{aligned} & \sum_{d \geq 0} \sum_{\delta \geq 0} \frac{t^d q^{d(d+3)/2 - \delta}}{(d(d+3)/2 - \delta)!} N^{d, \delta}(y) \\ &= \sum_{d \geq 0} t^d \sum_{\delta \geq 0} \left\langle v_{\emptyset} \left| \text{Coeff}_{t^d} \left[\frac{(qH_1(t))^{d(d+3)/2 - \delta}}{(d(d+3)/2 - \delta)!} \right] \right| \exp(a_{-1})v_{\emptyset} \right\rangle \\ &= \sum_{n \geq 0} \sum_{d \geq 0} t^d \left\langle v_{\emptyset} \left| \text{Coeff}_{t^d} \left[\frac{(qH_1(t))^n}{n!} \right] \right| \exp(a_{-1})v_{\emptyset} \right\rangle = \langle \exp(qH_1(t)) \exp(a_{-1}) \rangle. \end{aligned}$$

(2) is similar. Using again Corollary 3.7 and Remark 3.8 we have in the same way as before

$$\begin{aligned} & \sum_{c \geq 0} \sum_{d \geq 0} \sum_{\delta \geq 0} \frac{s^c t^d q^{\binom{d+1}{2}m + cd + c + d - \delta}}{(\binom{d+1}{2}m + cd + c + d - \delta)!} N^{(\Sigma_m, cF + dH), \delta}(y) \\ &= \sum_{c \geq 0} \sum_{d \geq 0} t^d \sum_{\delta \geq 0} \left\langle v_{(1^c), \emptyset} s^c \left| \text{Coeff}_{t^d} \left[\frac{(qH_m(t))^{\binom{d+1}{2}m + cd + c + d - \delta}}{(\binom{d+1}{2}m + cd + c + d - \delta)!} \right] \right| \exp(a_{-1})v_{\emptyset} \right\rangle \\ &= \sum_{c \geq 0} \sum_{n \geq 0} \left\langle v_{(1^c), \emptyset} s^c \left| \frac{(qH_m(t))^n}{n!} \right| \exp(a_{-1})v_{\emptyset} \right\rangle \\ &= \langle \exp(a_{-1}s)v_{\emptyset} | \exp(qH_m(t)) | \exp(a_{-1})v_{\emptyset} \rangle = \langle \exp(a_1s) \exp(qH_m(t)) \exp(a_{-1}) \rangle, \end{aligned}$$

where in the last step we use that a_1 is the adjoint of a_{-1} .

(3) By Corollary 3.7 and using the same arguments as in (1) we get

$$\begin{aligned} & \sum_{d \geq 0} \sum_{\delta \geq 0} \frac{t^d q^{\binom{m}{2}d^2 + md - \delta}}{(\binom{m}{2}d^2 + md - \delta)!} N^{(\mathbb{P}(1, (m-1), m), dH), \delta}(y) \\ &= \sum_{d \geq 0} t^d \sum_{\delta \geq 0} \left\langle \text{Coeff}_{t^d} \left[\frac{(qG_{m-1}(t))^{\binom{m}{2}d^2 + md - \delta}}{(\binom{m}{2}d^2 + md - \delta)!} \right] \right\rangle \\ &= \sum_{d \geq 0} t^d \sum_{n \geq 0} \left\langle \text{Coeff}_{t^d} \left[\frac{(qG_{m-1}(t))^n}{n!} \right] \right\rangle = \langle \exp(qG_{m-1}(t)) \rangle. \end{aligned}$$

□

Remark 3.10. In [7], [2] irreducible refined Severi degrees $N_0^{(S, L), \delta}(y)$, $N_0^{\Delta, \delta}(y)$ are introduced and studied. They give a count of irreducible tropical curves:

$$N_0^{\Delta, \delta} = \sum_{(C, h)} \text{mult}(C, h, y),$$

where the sum is now over *irreducible* δ -nodal tropical curves through $|\Delta \cap \mathbb{Z}| - 1 - \delta$ tropically generic points. At $y = 1$ they specialize to the irreducible Severi degrees counting irreducible complex curves of degree Δ and at $y = -1$ they specialize to the irreducible Welschiger invariants, counting irreducible real curves.

Corollary 3.9 also provides a generating function for the $N_0^{(S,L),\delta}$, as follows. By [7], [2] we have the formula

$$\sum_{L,\delta} \frac{z^{\dim |L| - \delta}}{(\dim |L| - \delta)!} v^L N_0^{(S,L),\delta}(y) = \log \left(\sum_{L,\delta} \frac{z^{\dim |L| - \delta}}{(\dim |L| - \delta)!} v^L N^{(S,L),\delta}(y) \right),$$

Here $\{v^L\}_L$, are elements of the Novikov ring i.e. $v^{L_1} v^{L_2} = v^{L_1 + L_2}$. On \mathbb{P}^2 , $\mathbb{P}(1, 1, m)$, $\mathbb{P}(1, m-1, m)$, L runs through the nH with $n \geq 0$, and on Σ_m , L runs through $dH + cF$ with $c, d \geq 0$.

Combining this with Corollary 3.9 we get the formulas

$$\begin{aligned} \sum_{d \geq 0} \sum_{\delta \geq 0} \frac{t^d q^{d(d+3)/2 - \delta}}{(d(d+3)/2 - \delta)!} N_0^{d,\delta}(y) &= \log (\langle \exp(qH_1(t)) \exp(a_{-1}) \rangle), \\ \sum_{c \geq 0} \sum_{d \geq 0} \sum_{\delta \geq 0} \frac{s^c t^d q^{\binom{d+1}{2} m + cd + c + d - \delta}}{(\binom{d+1}{2} m + cd + c + d - \delta)!} N_0^{(\Sigma_m, cF + dH), \delta}(y) \\ &= \log (\langle \exp(a_1 s) \exp(qH_m(t)) \exp(a_{-1}) \rangle), \\ \sum_{d \geq 0} \sum_{\delta \geq 0} \frac{t^d q^{\frac{m}{2} d^2 + (\frac{m}{2} + 1)d - \delta}}{(\frac{m}{2} d^2 + (\frac{m}{2} + 1)d - \delta)!} N_0^{(\mathbb{P}(1,1,m), dH), \delta}(y) &= \log (\langle \exp(qH_m(t)) \exp(a_{-1}) \rangle), \\ \sum_{d \geq 0} \sum_{\delta \geq 0} \frac{t^d q^{\binom{m}{2} d^2 + md - \delta}}{(\binom{m}{2} d^2 + md - \delta)!} N_0^{(\mathbb{P}(1,(m-1),m), dH), \delta}(y) &= \log (\langle \exp(qG_{m-1}(t)) \rangle). \end{aligned}$$

Remark 3.11. Corollary 3.9 can be easily extended to relative refined Severi degrees. For partitions α, β , write $u^\alpha := \prod_i u_i^{\alpha_i}$, $w^\beta := \prod_i w_i^{\beta_i}$. Then we have the easy identity

$$\exp \left(\sum_{n > 0} \frac{1}{[n]_y} (b_{-n} u_n + a_{-n} w_n) \right) v_\emptyset = \sum_{\substack{\alpha, \beta \\ \text{partitions}}} \frac{v_{\beta, \alpha}}{I_y^{\alpha + \beta}} u^\alpha w^\beta.$$

Using this, the same arguments as in the proof of Corollary 3.9 show for instance

$$\begin{aligned} \sum_{d \geq 0} \sum_{\delta \geq 0} \sum_{\alpha, \beta} \frac{t^d q^{d(d+3)/2 - \delta - d + |\beta|} N^{d,\delta}(\alpha, \beta)(y) u^\alpha}{(d(d+3)/2 - \delta - d + |\beta|)!} \frac{w^\beta}{\alpha!} \\ = \left\langle \exp(qH_1(t)) \exp \left(\sum_{n > 0} \frac{u_n b_{-n} + w_n a_{-n}}{[n]_y} \right) \right\rangle. \end{aligned}$$

Remark 3.12. The refined Severi degrees $N^{\Delta, \delta}(y)$ and $N^{\Delta, \delta}(\alpha, \beta)(y)$ specialize to the Severi degrees $N^{\Delta, \delta}$ and $N^{\Delta, \delta}(\alpha, \beta)$ at $y = 1$ and to the tropical Welschinger invariants $W^{\Delta, \delta}$ and $W^{\Delta, \delta}(\alpha, \beta)$ at $y = -1$. We write $[\ , \]_1, \langle \ | \ \rangle_1$, (resp. $[\ , \]_{-1}, \langle \ | \ \rangle_{-1}$) for the specializations of $[\ , \]$ and $\langle \ | \ \rangle$ at $y = 1$ (resp. $y = -1$). Thus, to obtain the results for the Severi degrees we just have to specialize $y = 1$ respectively $y = -1$ in Theorem 3.5, Corollary 3.6, Corollary 3.7, Corollary 3.8. This is the same as replacing $[\ , \]$ by $[\ , \]_1$ respectively $[\ , \]_{-1}$ in the commutation relation of the Heisenberg algebra, and $\langle \ | \ \rangle$ by $\langle \ | \ \rangle_1$ respectively $\langle \ | \ \rangle_{-1}$.

- (1) As $[n]_1 = n$, we get for $y = 1$ the standard Heisenberg algebra modelled on the hyperbolic lattice with commutation relations $[a_n, a_m]_1 = 0 = [b_n, b_m]_1$, $[a_n, b_m]_1 = n\delta_{n,-m}$. and the inner product of the basis vectors F is $\langle v_{\mu,\nu} | v_{\mu',\nu'} \rangle_1 = \prod_i \frac{i^{\mu_i}}{\mu_i!} \prod_j \frac{j^{\nu_j}}{\nu_j!} \delta_{\mu,\nu'} \delta_{\mu',\nu}$.
- (2) For the specialization $y = -1$, we find that all a_n, b_n with n even lie in the center of the Heisenberg algebra. Therefore we will consider the Lie algebra \mathcal{H}^{odd} generated by the a_n, b_n , with n odd, with the commutation relations (for n, m odd)

$$[a_n, a_m]_{-1} = 0 = [b_n, b_m]_{-1}, \quad [a_n, b_m]_{-1} = (-1)^{(n-1)/2} \delta_{n,-m}.$$

The Fock space F^{odd} is generated by applying the creation operators in \mathcal{H}^{odd} to v_\emptyset . We call a partition $\mu = (1^{\mu_1}, 2^{\mu_2}, \dots)$ *odd* if $\mu_i = 0$ for i even. Then a basis of F^{odd} is given by the vectors $v_{\mu,\nu} = a_{-\mu} b_{-\nu} v_\emptyset$ with μ and ν odd, and the inner product of the basis vectors is given by

$$(3.5) \quad \langle v_{\mu,\nu} | v_{\mu',\nu'} \rangle_{-1} = \frac{(-1)^{(\|\mu\| + \|\nu\| - |\mu| - |\nu|)/2}}{\prod_i \mu_i! \prod_j \nu_j!} \delta_{\mu,\nu'} \delta_{\mu',\nu}.$$

If we restrict attention to the *absolute* Welschinger invariants $W^{(S,L),\delta}$, instead of the relative Welschinger invariants $W^{(S,L),\delta}(\alpha, \beta)$, we see that the right hand sides of the formulas of Theorem 3.5, Corollary 3.6, Corollary 3.7, are of the form $\langle v_{\mu,\nu} | A v_{\mu',\nu'} \rangle$ for some element $A \in \mathcal{H}$, and μ, ν satisfying $\|\mu\| - |\mu| = \|\nu\| - |\nu| = 0$. Therefore we can replace the inner product in (3.5) by the standard inner product on F^{odd}

$$(3.6) \quad \langle v_{\mu,\nu} | v_{\mu',\nu'} \rangle_* = \frac{\delta_{\mu,\nu'} \delta_{\mu',\nu}}{\prod_i \mu_i! \prod_j \nu_j!}.$$

For simplicity we only formulate the version for the Welschinger invariants of Corollary 3.7: Denote

$$\begin{aligned} H_m^{odd}(t) &:= \sum_{k>0 \text{ odd}} b_{-k} b_k + \sum_{\|\mu\|=\|\nu\|-m} a_{-\mu} a_{-\nu} \\ G_m^{odd}(t) &:= \sum_{k>0 \text{ odd}} b_{-k} b_k + \sum_{\|\mu\|=\|\nu\|+1} a_{-\mu} a_\nu + t \sum_{\|\mu\|=\|\nu\|-m} a_{-\mu} a_\nu. \end{aligned}$$

where the second (respectively second and third) sums are now over pairs of odd partitions. Then we have

$$\begin{aligned} W^{d,\delta} &= \left\langle v_\emptyset \left| \text{Coeff}_{t^d} [H_1^{odd}(t)^{d(d+3)/2-\delta}] \right| v_{(1^d),\emptyset} \right\rangle_*, \\ W^{(\Sigma_m, cF+dH),\delta}(y) &= \left\langle v_{(1^c),\emptyset} \left| \text{Coeff}_{t^d} [H_m^{odd}(t)^{\binom{d+1}{2}m+cd+c+d-\delta}] \right| v_{(1^{dm+c}),\emptyset} \right\rangle_*, \\ W^{(\mathbb{P}(1,(m-1),m),dH),\delta}(y) &= \left\langle \text{Coeff}_{t^d} [G_{m-1}^{odd}(t)^{\binom{m}{2}d^2+md-\delta}] \right\rangle_*. \end{aligned}$$

4. MARKED FLOOR DIAGRAMS

Let Δ be an h -transverse polygon given by $d_\Delta^b, d_\Delta^t \in \mathbb{Z}_{\geq 0}$ and multisets r_Δ, l_Δ with $|l_\Delta| = |r_\Delta|$, such that $d_\Delta^t + \|r_\Delta\| = d_\Delta^b + \|l_\Delta\|$ as in Section 2.3. Recall that $h_\Delta = |l_\Delta| = |r_\Delta|$ is the height of Δ .

Definition 4.1. A Δ -floor diagram \mathcal{D} consists of:

- two orderings R and L of r_Δ and l_Δ , and a sequence $(s_1, \dots, s_{h_\Delta})$ of non-negative integers such that $|s| = d_\Delta^t$,
- a graph on a vertex set $\{1, \dots, h_\Delta\}$ of *white vertices*, possibly with multiple edges, with edges directed $i \rightarrow j$ for $i < j$, and
- edge weights $w(e) \in \mathbb{Z}_{>0}$ for all edges e such that for every vertex j ,

$$\text{div}(j) := \sum_{\substack{\text{edges } e \\ j \xrightarrow{e} k}} w(e) - \sum_{\substack{\text{edges } e \\ i \xrightarrow{e} j}} w(e) \leq r_j - l_j + s_j.^1$$

We call $\mathbf{a} := (d^t, R - L)$ the *divergence sequence*.

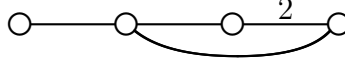


FIGURE 2. Floor diagram with $R = (1, 1, 1, 1)$, $L = (0, 0, 0, 0)$, and $S = (0, 0, 0, 0)$.

Definition 4.2. A *marked floor diagram* or *marking* is obtained from a floor diagram \mathcal{D} as follows:.

Step 1: For each vertex j of \mathcal{D} , create s_j new indistinguishable *black vertices* and connect them to j with new edges directed towards j .

Step 2: For each vertex j of \mathcal{D} , create $R_j - L_j + s_j - \text{div}(j)$ new indistinguishable vertices and connect them to j with new edges directed away from j . This makes the divergence of vertex j equal to $R_j - L_j$ for $1 \leq j \leq h_\Delta$.

Step 3: Subdivide each edge of the original floor diagram \mathcal{D} into two directed edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Denote the resulting graph $\tilde{\mathcal{D}}$.

Step 4: Linearly order the vertices of $\tilde{\mathcal{D}}$ extending the order of the vertices of the original floor diagram \mathcal{D} such that, as before, each edge is directed from a smaller vertex to a larger vertex.

The extended graph Γ together with the linear order on its vertices is called a *marked floor diagram*.

We need to count marked floor diagrams up to equivalence. Two such Γ_1, Γ_2 are *equivalent* if Γ_1 can be obtained from Γ_2 by permuting edges without changing their weights; *i.e.*, if there exists an automorphism of weighted graphs which preserves the vertices of \mathcal{D} and maps Γ_1 to Γ_2 .

¹This inequality will become clear when we define marked floor diagrams.

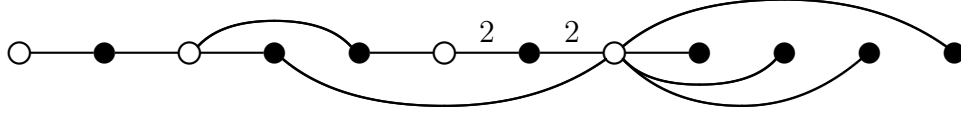


FIGURE 3. Marking for floor diagram of Figure 2.

If Γ is a marked floor diagram obtained from a Δ -floor diagram we label its k vertices $\{1, \dots, k\}$ left to right. The *cogenus* of Γ is $\delta(\Gamma) := \#\Delta - 1 - k$.

Our definition of the cogenus of a marked floor diagrams agrees with [6, 1]: let Γ be a marked Δ -floor diagram corresponding to a tropical curve C of degree Δ with cogenus $\delta(C)$ through a (vertically stretched, see [2, Definition 3.6]) tropical point configuration Π . Then $|\Pi| = \#\Delta - 1 - \delta(C)$. The vertices of Γ correspond to the points in Π , see [3, Section 5.2], so Γ has $\#\Delta - 1 - \delta(C)$ vertices. We defined the cogenus $\delta(\Gamma)$ precisely so that $\delta(\Gamma) = \delta(C)$.

The *refined multiplicity* of Γ is

$$\text{mult}(\Gamma, y) := \prod_{\text{edges } e \text{ of } \Gamma} [w(e)]_y.$$

We can compute the refined Severi degrees $N^{\Delta, \delta}$, for h -transverse Δ , in terms of marked floor diagrams.

Theorem 4.3 ([1, Theorem 2.7]). *For any h -transverse polygon Δ and any $\delta \geq 0$:*

$$N^{\Delta, \delta} = \sum_{[\Gamma]} \text{mult}(\Gamma, y).$$

The sum is over equivalence classes $[\Gamma]$ of marked Δ -floor diagrams of cogenus δ .

In [1, Theorem 2.7] the formula is instead a sum of $\text{mult}(\mathcal{D}, y)\nu(\mathcal{D})$ over all floor diagrams \mathcal{D} of cogenus δ , where $\text{mult}(\mathcal{D})$ is the number of markings of \mathcal{D} up to isomorphism, but this is clearly equivalent. Note that if Γ is a marking of \mathcal{D} , then $\text{mult}(\Gamma, y) = \text{mult}(\mathcal{D}, y)$, because all the inner edges of \mathcal{D} (which are the only ones with multiplicity different from 1) are divided in Γ into two edges.

Remark 4.4. We find the following explicit description of marked Δ -floor diagrams. A marked Δ -floor diagram Γ of cogenus δ is a directed colored graph with vertex set $\{1, \dots, \#\Delta - 1 - \delta\}$. All edges are directed $i \rightarrow j$ with $i < j$. The vertices i which lie only on edges $i \rightarrow j$ with $i < j$ are called *source vertices*, and those which lie only on edges $j \rightarrow i$ are called *sink vertices*. The diagrams are required to satisfy the following:

- (1) There are h_Δ white vertices $\{i_1, \dots, i_{h_\Delta}\}$, the other vertices are black, all edges connect vertices of different colors.
- (2) A black vertex that is not a source or sink vertex has precisely one incoming and one outgoing edge, both of the same weight.
- (3) There are precisely d_Δ^t black source vertices and d_Δ^b black sink vertices.
- (4) All the edges connected to black source or sink vertices have weight 1.

- (5) There are orderings $R = (R_1, \dots, R_{h_\Delta})$ and $L = (L_1, \dots, L_{h_\Delta})$ of r_Δ and l_Δ , such that for every white vertex i_j we have

$$\operatorname{div}(i_j) := \sum_{\substack{\text{edges } e \\ i_j \xrightarrow{e} k}} w(e) - \sum_{\substack{\text{edges } e \\ i \xrightarrow{e} i_j}} w(e) = R_j - L_j.$$

The equivalence relation is by permuting the edges of the same weight, leaving all the white vertices fixed.

As in the case of tropical curves earlier, there is a relative version of marked floor diagrams. Let α and β be two sequences with $\|\alpha\| + \|\beta\| = d_\Delta^b$.

Definition 4.5. An (α, β) -marked floor diagram or (α, β) -marking of a floor diagram \mathcal{D} is defined as follows:

Step 1: As Step 1 in Definition 4.2.

Step 2: Fix a pair of collections of sequences $(\{\alpha^i\}, \{\beta^i\})$, where i runs over the vertices of \mathcal{D} , with:

- (1) The sums over each collection satisfy $\sum_i \alpha^i = \alpha$ and $\sum_i \beta^i = \beta$.
- (2) For all vertices i of \mathcal{D} we have $\sum_{j \geq 1} j(\alpha_j^i + \beta_j^i) = R_i - L_i + s_i - \operatorname{div}(i)$.

The second condition says that the “degree of the pair (α^i, β^i) ” is compatible with the divergence at vertex i . Each such pair $(\{\alpha^i\}, \{\beta^i\})$ is called *compatible* with \mathcal{D} and (α, β) .

Step 3: For each vertex i of \mathcal{D} and every $j \geq 1$ create β_j^i new vertices, called β -vertices and illustrated as \bullet , and connect them to i with new edges of weight j directed away from i . For each vertex i of \mathcal{D} and every $j \geq 1$ create α_j^i new vertices, called α -vertices and illustrated as \odot , and connect them to i with new edges of weight j directed away from i .

Step 4: Subdivide each edge of the original floor diagram \mathcal{D} into two directed edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Denote the resulting graph $\tilde{\mathcal{D}}$.

Step 5: Linearly order the vertices of $\tilde{\mathcal{D}}$ extending the order of the vertices of the original floor diagram \mathcal{D} such that, as before, each edge is directed from a smaller vertex to a larger vertex. Furthermore, we require that the α -vertices are largest among all vertices, and for every pair of α -vertices $i' > i$, the weight of the i' -adjacent edge is larger than or equal to the weight of the i -adjacent edge.

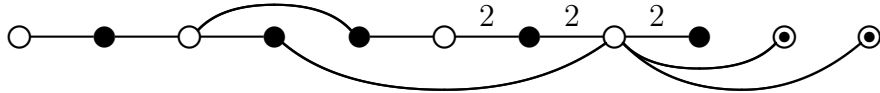


FIGURE 4. $(1^2, 2^1)$ -marking for floor diagram of Figure 2.

The extended graph Γ together with the linear order on its vertices is called an (α, β) -marking of the original floor diagram \mathcal{D} , and the diagram obtained this way is called an (α, β) -marked Δ -floor diagram. We count (α, β) -marked floor diagrams up

to equivalence. Two (α, β) -marked Δ -floor diagrams Γ_1, Γ_2 are *equivalent* if there exists a weight preserving isomorphism of weighted graphs mapping Γ_1 to Γ_2 which fixes the white vertices. The *refined multiplicity* of Γ is

$$\text{mult}(\Gamma, y) = \prod_e [w(e)]_y$$

where the product ranges over all edges e of Γ excluding edges adjacent to α -vertices.

The following theorem can be proved by combining the argument of the proofs of [2, Proposition 7.7] ($S = \mathbb{P}^2$ with tangency conditions) and [2, Theorem 5.7] ($S = S(\Delta)$ without tangency conditions) in a straightforward way.

Theorem 4.6. *For any h -transverse polygon Δ , any $\delta \geq 0$, and any pair of sequences α and β with $\|\alpha\| + \|\beta\| = d_\Delta^b$:*

$$N^{\Delta, \delta}(\alpha, \beta)(y) = \sum_{[\Gamma]} \text{mult}(\Gamma, y),$$

with the sum over the equivalence classes of (α, β) -marked Δ -floor diagrams.

For the comparison with Feynman graphs, we want to consider a slightly modified version of the (α, β) -marked Δ -floor diagrams, which we call *extended* (α, β) -marked Δ -floor diagrams:

Step 6: Given an (α, β) -marked Δ -floor diagram Γ , with vertices $\{1, \dots, k\}$ we define its extension $\hat{\Gamma}$, by adding a white vertex 0 with d_Δ^t outgoing edges of weight 1 and connecting it to the d_Δ^t black source vertices of Γ . Furthermore we add a white vertex $k+1$ and connect it to the $|\beta|$ black sink vertices l_i of Γ , each by an edge of the same weight as the edge ending in l_i . Finally we make the α -vertices black.

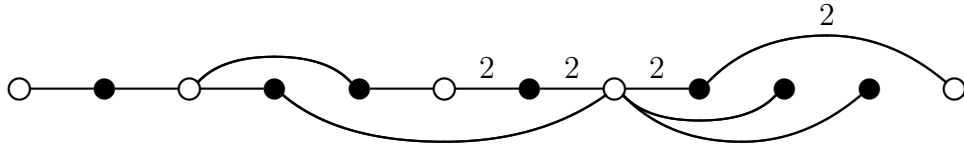


FIGURE 5. Extension of Figure 4. Note that this diagram is also a Feynman diagram for $a_{(1)}b_{-1}b_1a_{-(1)}a_{(1^2)}(b_{-1}b_1)^2a_{-(1)}a_{(2)}b_{-2}b_2a_{-(1,2)}a_{(1^2,2)}b_{-2}b_2(b_{-1})^2a_{-(2)}$ (see below).

Remark 4.7. We find the following explicit description of extended (α, β) -marked Δ -floor diagrams. An (α, β) -marked Δ -floor diagram of cogenus δ is a directed colored weighted graph Γ with vertex set $\{0, \dots, t+1\}$ with $t = \#\Delta - 1 - \delta - \|\alpha\| - \|\beta\| + |\alpha| + |\beta|$, we also let $s := t - |\alpha|$ and for each $l \geq 0$ let $s_l = s + \sum_{j \leq l} \alpha_j$. All edges are directed $i \rightarrow j$ with $i < j$. The diagrams are required to satisfy the following:

- (1) There are indices $0 < i_1 < \dots < i_{h_\Delta} \leq s$, such that the white vertices are $\{0, i_1, \dots, i_{h_\Delta}, t+1\}$. The other vertices are black. $0, s+1, \dots, t+1$, are sink vertices.

- (2) All edges connect vertices of different colors.
- (3) Each black vertex, with the exception of the sink vertices, has precisely one incoming and one outgoing edge of the same weight.
- (4) The edges connected to $0, s+1, \dots, t+1$ are the following:
 - (a) 0 has d_Δ^t outgoing edges, all of weight 1.
 - (b) For $i = s_k + 1, \dots, s_{k+1}$ the vertex i has one incoming edge of weight k .
 - (c) $t+1$ has β_i incoming edges of weight i for all i .
- (5) There are orderings $R = (R_1, \dots, R_{h_\Delta})$ and $L = (L_1, \dots, L_{h_\Delta})$ of r_Δ and l_Δ , such that for every white vertex $i_j \notin \{0, t+2\}$ we have

$$\text{div}(i_j) := \sum_{\substack{\text{edges } e \\ i_j \xrightarrow{e} k}} w(e) - \sum_{\substack{\text{edges } e \\ i \xrightarrow{e} i_j}} w(e) = R_j - L_j.$$

An equivalence $\widehat{\Gamma}_1 \rightarrow \widehat{\Gamma}_2$ of extended (α, β) -marked Δ -floor diagrams is an isomorphism of weighted directed graphs by permutation of the edges, fixing the white vertices. We also put

$$\text{mult}(\widehat{\Gamma}, y) := \prod_{e \text{ edges of } \widehat{\Gamma}} [w(e)]_y.$$

Corollary 4.8.

$$N^{\Delta, \delta}(\alpha, \beta) = \frac{1}{I_y^{\alpha+\beta}} \sum_{[\widehat{\Gamma}]} \text{mult}(\widehat{\Gamma}, y).$$

The sum is over equivalence classes of extended (α, β) -marked Δ -floor diagrams.

Proof. The result follows from the following easy facts: (1) (α, β) -marked floor diagram and extended (α, β) -marked floor diagram are obviously in bijection, and the same holds for the equivalence classes. (2) By definition, if Γ is an (α, β) -marked floor diagram, and $\widehat{\Gamma}$ is its extension, then by definition

$$\text{mult}(\widehat{\Gamma}, y) = \text{mult}(\Gamma, y) \prod_i ([i]_y^{\alpha_i + \beta_i}).$$

□

5. FEYNMAN DIAGRAMS

We associate Feynman diagrams to monomials $M = m_0 \cdots m_l$, where each m_i is of the form $a_{-\mu}a_\nu$ or $b_{-k}b_k$ or b_{-k} for partitions μ, ν and positive integers k .

Definition 5.1. A *Feynman diagram* for $M = m_0 \cdots m_l$ as above is a directed weighted graph with vertices $0, \dots, l$. If $m_i = a_{-\mu}a_\nu$ then the vertex i is white with $|\mu|$ incoming edges with μ_j of weight j for all j and $|\nu|$ outgoing edges with ν_j of weight j for all j . If $m_i = b_{-k}b_k$, then the vertex i is black with one incoming and one outgoing edge of both weight k . If $m_i = b_{-k}$, then the vertex i is black with one incoming edge of weight k . All edges are directed $i \rightarrow j$ with $i < j$ and they connect vertices of different color. An equivalence $\Gamma_1 \rightarrow \Gamma_2$ of Feynman diagrams for M is an isomorphism of weighted directed colored graphs, by permutation of the edges of the

same weight, leaving the white vertices fixed. For an edge e of a Feynman diagram we denote its weight by $w(e)$. For a Feynman diagram Γ , its multiplicity is

$$\text{mult}(\Gamma, y) := \prod_{e \text{ edges of } \Gamma} [w(e)]_y.$$

The following can be viewed as a version of the classical Wick's theorem [11].

Proposition 5.2 (Wick's theorem). *Let $M = m_1 \cdots m_l$ be a monomial in the $a_{-\mu} a_\nu$, $b_{-k} b_k$, b_{-k} . Then*

$$\langle M \rangle = \sum_{[\Gamma]} \text{mult}(\Gamma, y),$$

where the sum runs over all equivalence classes of Feynman diagrams for M .

Proof. To simplify notations we will write $a[0]_i := a_i$, $a[1]_i := b_i$. Note that the commutation relations (2.1) are

$$[a[s]_i, a[t]_j] = [i]_y \delta_{i,-j} \delta_{s,1-t}, \quad s, t \in \{0, 1\}.$$

(1) Now let $N = a[s_1]_{i_1} \cdots a[s_l]_{i_l}$ be any monomial in the $a[s]_i$, $i \in \mathbb{Z}_{\neq 0}$, $s \in \{0, 1\}$. We compute $\langle N \rangle$. If $i_1 \leq 0$, then

$$\langle N \rangle = \langle a[s_1]_{-i_1} v_\emptyset \mid a[s_2]_{i_2} \cdots a[s_l]_{i_l} v_\emptyset \rangle = 0,$$

because $a[s_1]_{-i_1} v_\emptyset = 0$. If $i_1 > 0$, applying the commutation relation, we get

$$\langle N \rangle = \sum_{\{m \mid i_m = -i_1, s_m = 1 - s_1\}} [i_1]_y \langle a[s_2]_{i_2} \cdots \widehat{a[s_m]_{i_m}} \cdots a[s_l]_{i_l} \rangle,$$

where the $\widehat{}$ means that the factor is removed. By induction this gives that $\langle N \rangle = 0$, if l is odd, and if $l = 2m$, then

$$(5.1) \quad \langle N \rangle = \sum_{I_1, \dots, I_m} \prod_{j=1}^m [w(I_j)]_y.$$

Here the sum is over all decompositions of $\{1, \dots, 2m\}$ into disjoint subsets of 2 elements $I_j = \{I_j^1, I_j^2\}$, with the following properties:

$$I_j^1 < I_j^2, \quad i_{I_j^1} > 0, \quad i_{I_j^2} = -i_{I_j^1}, \quad s_{I_j^2} = 1 - s_{I_j^1},$$

and we write $w(I_j) = i_{I_j^1}$.

We can view this as a count of directed graphs with multiplicities. For each factor $a[s_m]_{i_m}$ we place a vertex at the point m . The vertex is white if $s_m = 0$ and black if $s_m = 1$. It has one incoming edge of weight $-i_m$ if $i_m < 0$, it has one outgoing edge of weight i_m if $i_m > 0$, and no further edges. A graph Γ for $\langle N \rangle$ is a directed graph with these vertices and edges, so that every edge contains two vertices, and every vertex is connected by an edge to one other of a different color. We denote $w(e)$ the weight of the edges of Γ . The multiplicity of Γ is

$$\text{mult}(\Gamma, y) = \prod_{e \text{ edge of } \Gamma} [w(e)]_y.$$

Clearly there is a bijection between the decomposition of $1, \dots, l$ into disjoint pairs of integers as in (5.1) and the graphs for $\langle N \rangle$. Thus by (5.1) we have

$$\langle N \rangle = \sum_{\Gamma \text{ graph for } N} \text{mult}(\Gamma, y).$$

(2) Now let again $M = m_1 \cdots m_l$ be a monomial in the $(a_{-\mu} a_{\nu})$, $(b_{-k} b_k)$, b_{-k} . Assume that the factors of the form $(a_{-\mu} a_{\nu})$ are m_{i_1}, \dots, m_{i_n} . Then write $m_{i_s} = a_{-\mu^s} a_{\nu^s}$ for $s = 1, \dots, n$, for partitions $\mu^s = (1^{\mu_1^s}, 2^{\mu_2^s}, \dots)$, $\nu^s = (1^{\nu_1^s}, 2^{\nu_2^s}, \dots)$. Then $M = \frac{1}{\prod_{s=1}^n \mu^s! \nu^s!} N$, where $N = \prod_{i=1}^l \tilde{m}_i$, with $\tilde{m}_i = m_i$ for $i \notin \{i_1, \dots, i_n\}$ and $\tilde{m}_{i_s} = \left(\prod_j a_{-j}^{\mu_j^s} \right) \left(\prod_j a_j^{\nu_j^s} \right)$. Thus

$$\langle M \rangle = \frac{1}{\prod_{s=1}^n \mu^s! \nu^s!} \sum_{\Gamma} \text{mult}(\Gamma, y),$$

where Γ runs through the diagrams for $\langle N \rangle$.

Finally we need to relate the Feynman diagrams for N to those for M . We obtain all the Feynman diagrams for M from the diagrams for N by replacing all the vertices corresponding to one factor $(a_{-\mu^s} a_{\nu^s})$ by one white vertex, and replacing the two vertices corresponding to a factor $b_{-k} b_k$ by one black vertex, and keeping all the edges and their multiplicities and orientations, and have an edge connected to a vertex in M if it was connected to one of the vertices in N corresponding to it. Under this operation the $\prod_{s=1}^n \mu^s! \nu^s!$ graphs corresponding to the reorderings of the factors in each of the factors $\prod_j (a_{-j})^{\mu_j^s} \cdot \prod_j (a_j)^{\nu_j^s}$ are mapped to equivalent Feynman diagrams for M (and two diagrams are mapped to equivalent Feynman diagrams if and only if they are related by such a reordering). The multiplicities of the graphs are preserved. Thus the claim follows. \square

Proof of Theorem 3.5. It is enough to prove (3.2), because (3.1) is a special case.

Let $t := \#\Delta - \delta - 1 - \|\alpha\| - \|\beta\| + |\alpha| + |\beta|$ and $s = t - |\alpha|$. We can write $I_y^{\alpha+\beta}$ times the right hand side of (3.2) as

$$(5.2) \quad \alpha! \left\langle a_{(1^{d_\Delta^t})} \left(\sum_{R,L} \text{Coeff}[H(T)^s] \right) b_{-\alpha} a_{-\beta} \right\rangle = \left\langle a_{(1^{d_\Delta^t})} \left(\sum_{R,L} \text{Coeff}[H(T)^s] \right) \left(\prod_i b_{-i}^{\alpha_i} \right) a_{-\beta} \right\rangle,$$

with R, L running through the orderings of r_Δ and l_Δ . Thus in order to prove the theorem it is enough to show:

Claim: *The Feynman diagrams for the right hand side of (5.2) are precisely the extended (α, β) -marked Δ -floor diagrams.*

Fix $R = (R_1, \dots, R_{h_\Delta})$, $L = (L_1, \dots, L_{h_\Delta})$. Then by definition

$$a_{(1^{d_\Delta^t})} \text{Coeff}[H(T)^s] \left(\prod_i b_{-i}^{\alpha_i} \right) a_{-\beta}$$

is the sum over all monomials $m_0 \cdots m_{t+1}$, satisfying the following:

$$(1) \quad m_0 = a_{(1^{d_\Delta^t})}$$

- (2) There exist indices $0 < i_1 < \dots < i_{h_\Delta} \leq s$ such that $m_{i_j} = a_{-\mu^j} a_{\nu^j}$ for partitions μ^j, ν^j with $\|\nu^j\| - \|\mu^j\| = R_j - L_j$ for $j = 1, \dots, h_\Delta$.
- (3) The other m_l with $l \leq s$ are of the form $b_{-k_l} b_{k_l}$ for some k_l .
- (4) For all $l \geq 0$ let $s_l = s + \sum_{i \leq l} \alpha_i$. Then $m_i = b_{-i}$ for all $i \in \{s_l + 1, \dots, s_{l+1}\}$.
- (5) $m_{t+1} = a_{-\beta}$.

Now by Definition 5.1 the Feynman diagrams up to equivalence for these monomials are precisely the extended (α, β) -marked Δ -floor diagrams up to equivalence, as described in Remark 4.7. \square

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